FUNDAMENTALS OF QUANTITATIVE MODELING

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Module 3: Probabilistic models
Module 3 content

• What are probabilistic models?
• Random variables and probability distributions -- the building blocks
• Examples of probabilistic models
• Summaries of probability distributions: means, variances and standard deviation
• Special random variables: Bernoulli, Binomial and Normal
• The Empirical Rule
Probabilistic models

• These are models that incorporate *random variables* and *probability distributions*

• Random variables represent the potential outcomes of an uncertain event

• Probability distributions assign probabilities to the various potential outcomes

• We use probabilistic models in practice because realistic decision making often necessitates recognizing uncertainty (in the inputs and outputs of a process)
Key features of a probabilistic model

- By incorporating *uncertainty* explicitly in the model we can measure the uncertainty associated with the outputs, for example by giving a range to a forecast, which is a more realistic goal.
- In a business setting incorporating *uncertainty* is synonymous with understanding and quantifying the *risk* in a business process, and ideally leads to better management decisions.
Oil prices

If you run an energy intensive business, an airline for example, then the price of oil is a key determinant of your profitability.

For medium or long-term investment planning (buying new planes) the future price of oil is an important consideration.

But who knows the price of oil in ten years? No-one. But we may be able to put a probability distribution around the future price and incorporate the uncertainty into the decision making process.
Valuing a drug development company

- A company has 10 drugs in a development portfolio
- Given a drug has been approved, you have predicted its revenue
- But whether a drug is approved or not is an uncertain future event (a random variable). You have estimated the probability of approval
- You only wish to invest in the company if the company’s expected total revenue for the portfolio is over $10B in 5 years time
- You need to calculate the *probability distribution* of the total revenue to understand the investment risk
Some examples of probabilistic models

• *Regression models* (module 4)
• *Probability trees*
• *Monte Carlo* simulation
• *Markov models*
Regression models

- \( E(Price \mid Carats) = -259.6 + 3721 \times Carats \)
- The gray band gives a prediction interval for the price of a diamond taken from this population.
Regression models

• Regression models use data to estimate the relationship between the mean value of the outcome ($Y$) and a predictor variable ($X$)

• The intrinsic variation in the raw data is incorporated into forecasts from the regression model

• The less noise in the underlying data the more precise the forecasts from the regression model will be
Probability trees

- Probability trees allow you to propagate probabilities through a sequence of events

\[ P(\text{Stop infringing}) = 0.1 + 0.9 \times 0.15 + 0.9 \times 0.85 \times 0.2 = 0.388. \]
Monte Carlo simulation

• From the demand model:
  
  Quantity = 60,000 Price^{-2.5}

• The optimal price was $p_{opt} = \frac{c \cdot b}{1+b}$, where $b = -2.5$, $c$ is the cost, $c = 2$, and $p_{opt} \approx 3.33$

• But what if $b$ is not known exactly?

• Monte Carlo simulation replaces the number -2.5 with a random variable, and recalculates $p_{opt}$ using different realizations of this random variable from some stated probability distribution
Input and output from a MC simulation

- Input: $b$ from a uniform distribution between -2.9 and -2.1
- Output: $p_{opt} = \frac{c}{1+b}$
- 100,000 replications
- Interval = (3.1, 3.7)
Markov chain models

• Dynamic models for discrete time state space transitions
• Example: employment status (the state of the chain)
• Treat time in 6 month blocks
• Model states:
  1. Employed
  2. Unemployed and looking
  3. Unemployed and not looking
Probability transition matrix

Markov property (lack of memory): transition probabilities only depend on the current state, not on prior states. Given the present, the future does not depend on the past.
Building blocks of probability models

- Random variables (discrete and continuous)
- Probability distributions
- Random variables represent the potential outcomes of an uncertain event
- Probability distributions assign probabilities to the various potential outcomes
A discrete random variable

• Roll a fair die

<table>
<thead>
<tr>
<th>X</th>
<th>1/6</th>
<th>1/6</th>
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<th>1/6</th>
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</tr>
</thead>
<tbody>
<tr>
<td>P(X = x)</td>
<td>1/6</td>
<td>1/6</td>
<td>1/6</td>
<td>1/6</td>
<td>1/6</td>
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</tbody>
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• Probabilities lie between 0 and 1 inclusive
• Probabilities add to 1
A continuous random variable

- The *percent change* in the S&P500 stock index tomorrow: $100 \times \frac{p_{t+1} - p_t}{p_t}$ where $p_t$ is the closing price on day $t$
- It can potentially take on *any* number between -100% and infinity
- For a continuous random variable probabilities are computed from areas under the *probability density function*
Probability distribution of S&P500 daily % change

Probability model for daily % change

Density

P(change < -0.5)
Key summaries of probability distributions

- Mean ($\mu$) measures centrality
- Two measures of spread:
  - Variance ($\sigma^2$)
  - Standard deviation ($\sigma$)
The Bernoulli distribution

• The random variable X takes on one of two values:
  – P(X = 1) = p
  – P(X = 0) = 1 - p

• Often viewed as an experiment that takes on two outcomes, success/failure. Success = 1 and failure = 0

• μ = E(X) = 1 × p + 0 × (1 - p) = p

• σ² = E(X - μ)² = (1 - p)² p + (0 - p)² (1 - p) = p(1 - p)

• σ = \sqrt{p(1 - p)}

• For p = 0.5, μ = 0.5, σ² = 0.25 and σ = 0.5
Example: drug development

- Will a drug under development be approved?
- $X = \begin{cases} 
  Yes = 1 \\
  No = 0 
\end{cases}$
- $P(X = Yes) = 0.65$
- $P(X = No) = 0.35$
- If drug is approved then the projected revenue is $500m$, 0 otherwise
- $\text{Expected(Revenue)} = 500m \times 0.65 + 0 \times 0.35 = 325m$
The Binomial distribution

- A Binomial random variable is the number of success in \( n \) independent Bernoulli trials.
- Independent means that \( P(A \text{ and } B) = P(A) \times P(B) \).
- Independence means that knowing that \( A \) has occurred provides no information about the occurrence of \( B \).
- Independence is a common simplifying assumption in many probability models and makes their construction and subsequent calculations much easier.
The Binomial distribution

- Example: toss a fair coin 10 times and count the number of heads (call this X)
- Then X has a Binomial distribution with parameters n = 10 and p = 0.5.
- In general: \( P(X = x) = \binom{n}{x} p^x (1 - p)^{n-x} \), where \( \binom{n}{x} \) is the binomial coefficient: \( \frac{n!}{x!(n-x)!} \)
- \( \mu = E(X) = np \), \( \sigma^2 = E(X - \mu)^2 = np(1 - p) \)
Binomial probability distributions

Binomial, $n = 10$, $p = 0.5$

Binomial, $n = 20$, $p = 0.75$
Example: Binomial models for markets (oil for example)

• Assume that the market either goes up or down each day
• It goes up $u\%$ with probability $p$ and down $d\%$ with probability $1-p$
• Assume days are independent
• Example: $p = 0.55$, $1 - p = 0.45$, $u = 0.25\%$, $d = 0.22\%$
• Take a time horizon of 3 days
• There are 8 possible outcomes:
  - \{UUU\}, \{UUD\}, \{UDU\}, \{UDD\}, \{DUU\}, \{DUD\}, \{DDU\}, \{DDD\}
• For each outcome there will be an associated market move. For example, if we see (U,U,U) then the market goes up by a factor of $1.0025 \times 1.0025 \times 1.0025 = 1.007519$, that is a little over $\frac{3}{4}$ of a percent.
Tree representation

Current price

Up
- Prob = 0.55 × 0.55 × 0.55 = 0.166, Move = 0.75%
- Prob = 0.55 × 0.55 × 0.45 = 0.136, Move = 0.28%
- Prob = 0.55 × 0.45 × 0.55 = 0.136, Move = 0.28%
- Prob = 0.55 × 0.45 × 0.45 = 0.111, Move = -0.19%
- Prob = 0.45 × 0.55 × 0.55 = 0.136, Move = 0.28%
- Prob = 0.45 × 0.55 × 0.45 = 0.111, Move = -0.19%
- Prob = 0.45 × 0.45 × 0.55 = 0.111, Move = -0.19%
- Prob = 0.45 × 0.45 × 0.45 = 0.091, Move = -0.66%

Down
The Normal distribution

• The Normal distribution, colloquially known as the *Bell Curve*, is the most important modeling distribution

• Many disparate processes can be well *approximated* by Normal distributions

• There are mathematical theorems (the Central Limit Theorem) that tell us Normal distributions should be expected in many situations

• A Normal distribution is characterized by its mean $\mu$ and standard deviation $\sigma$. It is symmetric about its mean
Examples

• There is a *universality* to the Normal distribution
  – Biological: heights and weights
  – Financial: stock returns
  – Educational: exam scores
  – Manufacturing: the length of an automotive component

• It is therefore often used as a distributional assumption in Monte Carlo simulations (knowing the mean and standard deviation is enough to define a Normal distribution)
Plots of various Normal distributions

Parameters
- $\mu=0$, $\sigma=1$
- $\mu=2$, $\sigma=1$
- $\mu=-3$, $\sigma=2$
- $\mu=3$, $\sigma=3$
The Empirical Rule

• The Empirical Rule is a rule for calculating probabilities of events when the underlying distribution or observed data is approximately Normally distributed

• It states
  – There is an approximate 68% chance that an observation falls within one standard deviation from the mean
  – There is an approximate 95% chance that an observation falls within two standard deviations from the mean
  – There is an approximate 99.7% chance that an observation falls within three standard deviations from the mean
The Empirical Rule illustrated

The Empirical Rule

Density

\( \mu - 3\sigma \)  \( \mu - 2\sigma \)  \( \mu - \sigma \)  \( \mu \)  \( \mu + \sigma \)  \( \mu + 2\sigma \)  \( \mu + 3\sigma \)

68%  95%  99.7%
Empirical Rule example

- Assume that the daily return on Apple’s stock is approximately Normally distributed with mean \( \mu = 0.13\% \) and \( \sigma = 2.34\% \)

- What is the probability that tomorrow Apple’s stock price increases by more than 2.47%?

- Technique: count how many standard deviations 2.47% is away from the mean, 0.13%. Call this counter the z-score

\[
Z = \frac{2.47 - 0.13}{2.34} = 1
\]

- So, from the Empirical Rule the probability equals approximately 16%
Illustrating the answer

*Daily returns on Apple stock*

- **50%** within the range of -2.21 to 0.13
- **34%** within the range of 0.13 to 2.47
- **16%** within the range of 2.47 to 4.81
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